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Bose–Einstein condensation in the Rindler space

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ABSTRACT

Based on the Unruh effect, we calculate the critical acceleration of the Bose–Einstein condensation in a free complex scalar field at finite density in the Rindler space. Our model corresponds to an ideal gas performing constantly accelerating motion in a Minkowski space–time at zero-temperature, where the gas is composed of the complex scalar particles and it can be thought to be in a thermal-bath with the Unruh temperature. In the accelerating frame, the model will be in the Bose–Einstein condensation state at low acceleration; on the other hand, there will be no condensation at high acceleration by the thermal excitation brought into by the Unruh effect. Our critical acceleration is the one at which the Bose–Einstein condensation begins to appear in the accelerating frame when we decrease the acceleration gradually. To carry out the calculation, we assume that the critical acceleration is much larger than the mass of the particle.

1. Introduction

The Bose–Einstein condensation is getting a lot of attention recently as a quantum fluid for the test of the analogy between sound waves in quantum fluids and scalar field fluctuation in curved space–times [1]. Thanks to this analogy, we can expect new progress and insight which are difficult only in the gravitational analysis in the quantum gravity and cosmology. Further, the experiments of the quantum gravity and cosmology are difficult due to the required energy level and the scale of phenomena. However, the experiments of the condensed matters will be free from such problems to some extent. It is thought for these reasons that pseudo experiments of the gravity are possible in the quantum fluids through the analogy.

In order to understand the quantum phenomena in the gravity and cosmology such as the Hawking radiation [2] and particle creation [3,4], the Unruh effect [5–8] is important in terms of the role that the event-horizons play.

The Unruh effect is a prediction that one moving in the Minkowski space–time with a linear constant acceleration experiences the space–time as a thermal-bath with the Unruh temperature, $T_U = \hbar a/(2\pi c k_B) \approx 4 \times 10^{-23}\, a/(\mathrm{cm/s^2})\,[\mathrm{K}]$, where a is the acceleration.

Now, various experimental attempts in the condensed matters to observe the gravitational phenomena are being invented (see Ref. [9] for example). Particularly as for the experiments to detect the Unruh effect, there are attempts in Bose–Einstein condensates [11], graphenes [12] and Berry phases [13]. For other attempts see Ref. [14], for example, and related references.

We also address the issue of the Unruh effect in the Bose–Einstein condensation. Whether the Bose–Einstein condensation occurs or not is determined by temperature. We assume in this paper that the Unruh temperature exists in the constantly accelerating system according to the Unruh effect mentioned above. At this time we can think that the Unruh effect affects the Bose–Einstein condensation. Although an enormous number of studies have been done on the Unruh effect and the Bose–Einstein condensation so far, these are performed separately and little is known about the Unruh effect in the Bose–Einstein condensation at this moment. Since both the Unruh effect and the Bose–Einstein condensation are very important in the fundamental physics, new understandings could be expected by combining the Unruh effect and the Bose–Einstein condensation. In this paper, from such a background, we calculate the critical acceleration for the Bose–Einstein condensation based on the thermal excitation brought into by the Unruh effect.

Let us here explain the Bose–Einstein condensation briefly (for more details, see Ref. [10] for example). The Bose–Einstein condensation state is the situation that all the particle stay in the least energy state uniformly owing to the Bose–Einstein statistics, and it appears as a phase transition that all the particles uniformly drop to the least energy state as the entropically-favored state due to the Bose–Einstein statistics at some time. (In the Bose–Einstein

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condensation state, since all the particles are in a state such as the least energy state, de Broglie wave of each particle becomes longer and even, and eventually the system itself becomes a de Broglie wave.) The situation that the effect of the Bose–Einstein statistics is dominant can be considered as the low temperature region. Actually it is considered at the low temperature region that the particle's thermal motion energy is extremely small so that it does not excite the state of the particles from the least energy state. Hence, the Bose–Einstein condensation state emerges in the system composed of bosonic particle at extremely low temperature.

We calculate in this paper the critical acceleration at which the Bose–Einstein condensation begins to appear in the accelerating system by the Unruh effect when lowering the acceleration. We mention our critical acceleration more precisely. We first consider an ideal gas in the Minkowski space–time at zero-temperature, where the gas is composed of particles described by a free complex scalar field at finite density. As the gas is now in zero-temperature, it can be considered to be in the Bose–Einstein condensation state. We then start to accelerate such an ideal gas uniformly. At this time, according to the Unruh effect, since each constantly accelerating particle composing the gas will experience the temperature $T_U = \hbar a/(2\pi c k_B)$ (a is the acceleration) in the accelerating frame, the ideal gas performing uniformly accelerating motion will experience the temperature T_U as a whole in the accelerating frame. Here, what the gas experiences thermal means that the space which the gas observes is filled with some medium that are performing thermal fluctuation. In this paper, we consider that there is no interaction between the medium and the particles composing the gas without the thermal excitation. Hence it is considered that, as growing the acceleration gradually from lower acceleration, in the accelerating frame, the dissolution of the Bose–Einstein condensation is observed eventually, and finally the Bose–Einstein condensation disappears entirely at some acceleration. The critical acceleration we calculate is the acceleration at that time.

We here mention the emergence mechanism of the Bose–Einstein condensation in our model and its critical moment. It is composed of three steps. We first calculate the effective potential of the field meaning the particles composing the gas, and then obtain the particle density by performing a derivative to it with regard to the particle's chemical potential. Then we find that, if we decrease the acceleration with fixing the particle density to constant, either the particle's chemical potential or the absolute value of the zero-mode of the field has to grow. Here, the acceleration plays the role of the temperature in the accelerating frame as mentioned in the above paragraph, and the zero-mode of the field can be considered to correspond to the least energy state. Hence the absolute value of the zero-mode is considered as the expectation value of the Bose–Einstein condensation state, and whether it is zero or not corresponds to its disappearance or appearance, respectively. We consider to start with the high acceleration situation where there is no Bose–Einstein condensation in the accelerating frame, and decrease the acceleration gradually. At this time, we keep the particle density to constant. Namely, the number of particle is always constant in our model. Then, corresponding to no Bose–Einstein condensation, the absolute value of the zero-mode should be zero while we decrease acceleration. Thus the chemical potential must grow to keep the particle density to constant (step 1). However we find that there is an upper bound for the value of the chemical potential, which is need to avoid a divergence of the probability amplitude. Hence, when the chemical potential reaches the upper bound (step 2), there is no way but the absolute value of the zero-mode starts to grow (step 3), if we further decrease acceleration with keeping the particle density to constant. Thus, the moment that the chemical potential reaches the upper bound is the critical moment that the Bose–Einstein condensation appears, and this is the Bose–Einstein condensation in our model.

The situation where our analysis is actually performed is just before the Bose–Einstein condensation state starts to occur. To be more specific, our calculation to obtain the critical acceleration is performed in the situation that the absolute value of the zero-mode and the chemical potential are put to zero and the upper bound value respectively in the equation of the particle density as the critical moment mentioned above. Then the acceleration obtained from that equation is the critical acceleration.

In our analysis, due to some technical difficulty mentioned later, we consider the situation $m/a_c \ll 1$ (m and a_c are the mass of the particle and the critical acceleration, respectively), and take the leading contribution of this in our calculation, where m is the mass of the complex scalar particle considered in this paper and a_c is the critical acceleration. As the acceleration is proportional to the temperature in the relation given by the Unruh effect, this can be read as $k_B T_c/2 \gg mc^2$ (T_c is the critical temperature fixed with a_c), which is the relativistic situation that the particle's thermal motion energy is much higher than its static energy at the critical moment.

The result in this paper is hence applicable to the system of the complex scalar field with the critical temperature for the Bose–Einstein condensation that is much higher than the mass. Since the kind of the field considered in this paper is a complex scalar field composing an ideal gas, the actual particle that can correspond to this paper is some ideal complex scalar particle system whose critical temperature of the Bose–Einstein condensation locates in the region $k_B T/2 \gg mc^2$.

As stated above, in this study we perform the calculation for the critical acceleration in the Bose–Einstein condensation. So far several kinds of the critical accelerations for the spontaneous symmetry breaking induced by the Unruh effect have been carried out.

The critical acceleration for the chiral symmetry restoration was studied in Nambu–Jona-Lasinio model at zero and finite chemical potentials for quarks in Refs. [15] and [16] respectively. In Ref. [17] the critical acceleration for the restoration of the spontaneous symmetry breaking of the Z_2 symmetry in the real scalar field theory was studied.

There are also papers concluding that the Unruh effect does not contribute to the restoration of the spontaneous symmetry breaking [18]. The system in these papers is not the finite density, and their discussion is based on the effective potential. On the other hand, our Bose–Einstein condensation occurs from the relation between the chemical potential and the acceleration playing the role of the temperature. This is a different point between us and him. Because of this, their conclusion would not be applied to our study readily. We should take this issue to a future work.

Lastly, we introduce interesting papers that will have some connection to this study. Ref. [19] concludes that a larger acceleration should enhance a condensate as compared to those in a non-accelerated vacuum. In Ref. [20], although the background space–time is not the Rindler space, whether the Bose–Einstein condensation can occur or not is shown in an ideal boson gas model with a point-like impurity at finite temperature in some uniform gravitational force in each of $D = 1, 2$ and 3. Next, it is shown in Ref. [21] that free massless scalar particles are detected by a constantly accelerating detector with not the Bose–Einstein distribution but the Fermi–Dirac distribution in odd dimensions, and this problem is solved in Ref. [22].

2. The model

The model in this paper is a free complex scalar model at the finite density corresponding to an ideal gas in the constantly accelerating system with the acceleration a. The Lagrangian density is given as

$$\mathcal{L} = \hbar^2 g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - c^2m^2\phi^*\phi \tag{1}$$

with $\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ that means the field of the particle composing a gas, and μ, ν are the coordinate in the Rindler space explained below.

In the Minkowski space–time as an inertial system, the background space–time of a constantly accelerating system is given by the Rindler space. The Rindler coordinate in our notation is given as follows:

$$(\eta, \rho, y, z) \equiv (\eta, \rho, x_\perp) \quad \text{with} \quad x_\perp \equiv (y, z). \tag{2}$$

This relates with the Minkowski coordinate $(t, x, x_\perp)$ as

$$(t, x) = \frac{c}{a}\Big(\sinh\frac{at}{c},\, c\cosh\frac{at}{c}\Big) \equiv \rho\,(\sinh\eta,\, c\cosh\eta)\,, \tag{3}$$

where the accelerating direction has been thought to be in the x-direction. The Rindler metric in our notation is

$$ds^2 = (c\rho)^2 d\eta^2 - d(c\rho)^2 - dx_\perp^2. \tag{4}$$

In what follows, we use the unit system: $c = \hbar = k_B = 1$.

The constantly accelerating one in the Minkowski space–time corresponds to the one moving along a line on a constant ρ in the Rindler space. The relation between ρ and a are $\rho = 1/a$ as can be seen from eq. (3). Since constantly accelerating one experiences the system as a thermal-bath with the Unruh temperature T_U (the acceleration are in the relation: $T_U = a/2\pi$) by the Unruh effect, one moving along a line on a constant ρ in the Rindler space gets the temperature

$$T_U = 1/2\pi\rho. \tag{5}$$

Hence, the gas in our study is considered to be in such a thermal-bath with Unruh temperature T_U in the accelerating frame. Here, although our gas is considered to be in a thermal-bath with some medium performing thermal fluctuation giving the Unruh temperature T_U in the accelerating frame, we assume that there is no interaction between our gas and the medium other than the thermal excitation.

We can see from eq. (5) that varying ρ means varying the temperature. For this reason, how to interpret the results would be unclear if the four-dimensional space–time integration including the ρ-integration was performed.

Let us here turn to how this problem is handled in other papers on the critical acceleration given by the Unruh effect. In Refs. [15–17], the action is given with the four-dimensional space–time integration including the ρ-integration. This point is the problem. However, their calculation is once performed using the Green's function that has dependence on the ρ-direction, the coordinate ρ is treated as a constant in the calculation of the effective potential. As a result, the ρ-integration becomes just a volume factor in the calculation of the effective potential.

The difficulty in the treatment of the coordinate ρ is also mentioned at the chapter of conclusions and discussions in Ref. [23] in the context of the study of the Larmor radiation with the correction rooted in the Unruh effect.

In our analysis, as well as Refs. [15–17], we obtain the Green's function that has dependence on the ρ-direction. However, the space–time integration in our action is given without the ρ-integration. As a result, ρ is a parameter in our model.

We can see at this time that the action is needed to be multiplied by a quantity with the dimension of length so that the action becomes dimensionless. To this purpose, we put $d\rho$ in our action. This means that the process to obtain the path-integral representation given in eq. (6) for the probability amplitude from the operator formalism representation has been performed with fixing ρ. If the analysis in Refs. [15–17] is performed again by our way, there is no difference in the final result.

There are four regions separated by the event-horizons in the Rindler space. The region treated in this paper is the right wedge only.

3. The effective potential

3.1. Performance of the path-integral

We start with the probability amplitude:

$$Z = \int \mathcal{D}\pi_\eta\mathcal{D}\phi \times \exp\Big[i\int d^3x\, d\rho\,\gamma\,\Big(\pi^\eta\partial_\eta\phi + \pi^{\eta *}\partial_\eta\phi^* - (\mathcal{H} - \mu q)\Big)\Big], \tag{6}$$

where $\gamma \equiv \sqrt{-\det g_{\mu\nu}}$ and $d^3x \equiv d\eta\, d^2x_\perp$, and μ and q are the chemical potential and the density of particles, respectively. For the reason mentioned in Section 2, to fix the acceleration and Unruh temperature, the integration in the ρ-direction is not included in the space–time integration in the action. As a result, ρ is a parameter in our analysis, and the Unruh temperature that the gas experiences is given according to the relation in eq. (5). At this time, we can see that the action is needed to be multiplied by a quantity with the dimension of length so that it becomes dimensionless. We put $d\rho$ to this purpose. However the space–time integration is the one without the ρ-direction. Explanation for π^η and $\mathcal{H}$ is in what follows.

$(\pi^\mu, \pi^{\mu *}) \equiv \big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)},\, \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\big)$ are the momenta given as

$$(\pi^\eta, \pi^{\eta *}) = \big(g^{\eta\eta}\partial_\eta\phi^*,\, g^{\eta\eta}\partial_\eta\phi\big). \tag{7}$$

$\mathcal{H}$ is the Hamiltonian density given as

$$\begin{aligned}\mathcal{H} &= \pi^\eta\,\partial_\eta\phi + \pi^{\eta *}\,\partial_\eta\phi - \mathcal{L} \\ &= \pi^{\eta *}\pi_\eta - g^{ij}\partial_i\phi^*\partial_j\phi + m^2\phi^*\phi,\end{aligned} \tag{8}$$

where π^μ is defined as $\pi^\mu \equiv \frac{1}{\sqrt{2}}(\pi_1^\mu + i\pi_2^\mu)$. Correspondingly, the functional integration measure of π_η changes as $\mathcal{D}\pi_\eta = \mathcal{D}\pi_{1\eta}\mathcal{D}\pi_{2\eta}$. From eq. (7), we can see

$$\pi_{1,2}^\eta = g^{\eta\eta}\partial_\eta\phi_{1,2}. \tag{9}$$

It turns out that the conserved current associated with the U(1) global symmetry in our model is obtained as

$$J^\mu = -i\,g^{\mu\nu}(\phi\,\partial_\nu\phi^* - \phi^*\,\partial_\nu\phi). \tag{10}$$

Using this J^μ, the integral of conserved charge density can be written as

$$\int d^3x\,d\rho\,\gamma\,q = \int d^3x\,d\rho\,\gamma\,J^\eta = \int d^3x\,d\rho\,\gamma\,(-\pi_2^\eta\,\phi_1 + \pi_1^\eta\,\phi_1). \tag{11}$$

Substituting eqs. (8) and (11) into eq. (6), we can obtain the following Z:

$$Z = \int \mathcal{D}\pi_{1\eta}\mathcal{D}\pi_{2\eta}\mathcal{D}\phi \exp\Big[\frac{i}{2}\int d^3x\, d\rho\, \gamma\, \Big\{ - \frac{1}{2}\big(\pi_1^\eta \pi_{1\eta} - 2\,(\partial_\eta\phi_1 + \mu\,\phi_2)\,\pi_1^\eta\big) - \frac{1}{2}\big(\pi_2^\eta \pi_{2\eta} - 2\,(\partial_\eta\phi_2 + \mu\,\phi_1)\,\pi_2^\eta\big) + \frac{1}{2}\big((\partial_i\phi_1)^2 + (\partial_i\phi_2)^2 - m^2(\phi_1^2+\phi_2^2)\big)\Big\}\Big], \tag{12}$$

where $i, j = x_\perp$. Here, in the above, we perform the following rewritings:

$$\pi_1^\eta \pi_{1\eta} - 2\,(\partial_\eta\phi_1 + \mu\,\phi_2)\,\pi_1^\eta = g^{\eta\eta}\Big\{\big(\pi_{1\eta} - (\partial_\eta\phi_1 + \mu\,\phi_2)\big)^2 - (\partial_\eta\phi_1 + \mu\,\phi_2)^2\Big\}, \tag{13a}$$

$$\pi_2^\eta \pi_{2\eta} - 2\,(\partial_\eta\phi_2 + \mu\,\phi_1)\,\pi_2^\eta = g^{\eta\eta}\Big\{\big(\pi_{2\eta} - (\partial_\eta\phi_2 + \mu\,\phi_1)\big)^2 - (\partial_\eta\phi_2 + \mu\,\phi_1)^2\Big\}. \tag{13b}$$

Furthermore, we redefine the fields as

$$\pi_{1\eta} - (\partial_\eta\phi_1 + \mu\,\phi_2) \to \pi_{1\eta}, \tag{14a}$$

$$\pi_{2\eta} - (\partial_\eta\phi_2 + \mu\,\phi_1) \to \pi_{2\eta}. \tag{14b}$$

As a result, we can write Z as

$$Z = \mathcal{C}\int \mathcal{D}\phi \exp\Big[\frac{i}{2}\int d^3x\, d\rho\, \gamma\, \Big(g^{\eta\eta}(\partial_\eta\phi_1 + \mu\,\phi_2)^2 + (\partial_i\phi_1)^2 + g^{\eta\eta}(\partial_\eta\phi_2 + \mu\,\phi_1)^2 + (\partial_i\phi_2)^2 - m^2({\phi_1}^2 + {\phi_2}^2)\Big)\Big]. \tag{15}$$

with $\mathcal{C} \equiv \int \mathcal{D}\pi_{1\eta}\mathcal{D}\pi_{2\eta} \exp\Big[i\int d^3x\, d\rho\, \gamma\, g^{\eta\eta}\big((\pi_{1\eta})^2 + (\pi_{2\eta})^2\big)\Big]$. Performing the path-integral of $\pi_{1\eta}$ and $\pi_{2\eta}$ formally, we think that $\mathcal{C}$ become some factor. We ignore $\mathcal{C}$ in what follows. As a result, with some straight forward calculation, we can write Z in the following form,

$$Z = \int \mathcal{D}\phi \exp\Big[-\frac{i}{2}\int d^3x\, d\rho\, \gamma\, \Big(\phi_1 G\phi_1 + \phi_2 G\phi_2 + 2g^{\eta\eta}\mu\,(\phi_2\partial_\eta\phi_1 - \phi_1\partial_\eta\phi_2)\Big)\Big], \tag{16}$$

where $G \equiv \partial_\eta^2 + \gamma^{-1}g^{ij}\partial_i(\gamma\,\partial_j) + m^2 - g^{\eta\eta}\mu^2$.

Let us now rewrite the real and imaginary parts of the field into a convenient expression for the analysis of the Bose–Einstein condensation. As we have written in the introduction, the Bose–Einstein condensation can be considered as the situation that all particles get to be the least energy state. The least energy state can be considered as the zero-mode of the field in our model, and the situation that all the particles are in the least energy state can be considered as the condensation of the zero-mode. Hence, a convenient expression for the analysis of the Bose–Einstein condensation in our analysis is the one in which the zero-mode is separated as

$$\phi_1 \equiv \sqrt{2}\,\alpha\, \cos\theta + \hat{\phi}_1, \tag{17a}$$

$$\phi_2 \equiv \sqrt{2}\,\alpha\, \sin\theta + \hat{\phi}_2, \tag{17b}$$

where α plays the role of the expectation value of the condensation of the zero-mode, θ is a phase in the zero-mode and $\hat{\phi}_1$, and $\hat{\phi}_2$ are the non-zero modes. Depending on just before or after the condensation starts, α behaves as follows:

$\alpha = 0$: before the condensation

$\alpha \neq 0$: after the condensation

At this time, $\phi_1 G\,\phi_1$ and $\phi_2 G\,\phi_2$ can be written as

$$\phi_1 G\,\phi_1 = 2\alpha^2(m^2 - g^{\eta\eta}\mu^2)\,\cos^2\theta + \hat{\phi}_1 G\,\hat{\phi}_1, \tag{18}$$

$$\phi_2 G\,\phi_2 = 2\alpha^2(m^2 - g^{\eta\eta}\mu^2)\,\sin^2\theta + \hat{\phi}_2 G\,\hat{\phi}_2. \tag{19}$$

As a result, we can rewrite Z into

$$Z = \exp\Big[-i\,\alpha^2\int d^3x\, d\rho\, \gamma\,(m^2 - g^{\eta\eta}\mu^2)\Big]\int \mathcal{D}\hat{\phi}\exp\Big[-\frac{i}{2}\int d^3x\, d\rho\, \gamma\, (\hat{\phi}_1 \quad \hat{\phi}_2)\begin{pmatrix} G & -2g^{\eta\eta}\mu\,\partial_\eta \\ 2g^{\eta\eta}\mu\,\partial_\eta & G\end{pmatrix}\begin{pmatrix}\hat{\phi}_1 \\ \hat{\phi}_2\end{pmatrix} + \sqrt{2}\,\alpha\cos\theta\,\Big(G\,(\phi_1+\phi_2) + (\phi_1+\phi_2)\,G\Big)\Big]. \tag{20}$$

Our analysis will be performed just before the condensation as mentioned later. For this reason, we put $\alpha = 0$ in what follows. As a result, we can write Z as

$$Z = \int \mathcal{D}\hat{\phi}\exp\Big[-\frac{i}{2}\int d^3x\, d\rho\, \gamma\,(\hat{\phi}_1 \quad \hat{\phi}_2) \times \begin{pmatrix} G & -2g^{\eta\eta}\mu\,\partial_\eta \\ 2g^{\eta\eta}\mu\,\partial_\eta & G\end{pmatrix}\begin{pmatrix}\hat{\phi}_1 \\ \hat{\phi}_2\end{pmatrix}\Big]. \tag{21}$$

Then, we can see from the form of G given below eq. (16) that there should be the condition:

$$m^2 - g^{\eta\eta}\mu^2 \geq 0. \tag{22}$$

Otherwise the path-integrals of Z diverges at the configuration that all the momenta are zero. Further, the above relation gives the upper limit of the chemical potential for given a mass and an Unruh temperature.

Performing the diagonalization as

$$Z = \int \mathcal{D}\hat{\phi}\exp\Big[-\frac{i}{2}\int d^3x\, d\rho\, \gamma\,(\hat{\phi}_1 \quad \hat{\phi}_2)\,UU^{-1} \times \begin{pmatrix} G & -2g^{\eta\eta}\mu\,\partial_\eta \\ 2g^{\eta\eta}\mu\,\partial_\eta & G\end{pmatrix} UU^{-1}\begin{pmatrix}\hat{\phi}_1 \\ \hat{\phi}_2\end{pmatrix}\Big].$$

$$= \int \mathcal{D}'\hat{\phi}'\exp\Big[-\frac{i}{2}\int d^3x\, d\rho\, \gamma\,(\hat{\phi}'_1 \quad \hat{\phi}'_2) \times \begin{pmatrix} G + 2g^{\eta\eta}\mu\,\partial_\eta & 0 \\ 0 & G - 2g^{\eta\eta}\mu\,\partial_\eta\end{pmatrix}\begin{pmatrix}\hat{\phi}'_1 \\ \hat{\phi}'_2\end{pmatrix}\Big], \tag{23}$$

where $U \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1\end{pmatrix}$ is a unitary matrix defined to perform the above diagonalization, and correspondingly $\begin{pmatrix}\hat{\phi}'_1 \\ \hat{\phi}'_2\end{pmatrix} \equiv U^{-1}\begin{pmatrix}\hat{\phi}_1 \\ \hat{\phi}_2\end{pmatrix}$. At this transformation, the functional measure is also transformed, which we have described as $\mathcal{D}\hat{\phi} \to \mathcal{D}'\hat{\phi}'$. However, since U is a constant unitary matrix, the difference between $\mathcal{D}\hat{\phi}$ and $\mathcal{D}'\hat{\phi}'$ contribute only to some constant factor in the path-integral, and we ignore it in what follows.

Performing the path-integral, we can obtain

$$Z = \mathrm{Det}\Big(\big(G + 2g^{\eta\eta}\mu\,\partial_\eta\big)\big(G - 2g^{\eta\eta}\mu\,\partial_\eta\big)\Big)^{-1/2}. \tag{24}$$

In the above, with regard to the treatment of $d\rho$, we consider that the integration of this has been performed by assigning a value at one point. Hence, the effective action W defined as $Z = \exp iW$ can be written as

$$
\begin{aligned}
W &= \frac{i}{2}\,\mathrm{Log\,Det}\Big(\big(G + 2g^{\eta\eta}\mu\,\partial_\eta\big)\big(G - 2g^{\eta\eta}\mu\,\partial_\eta\big)\Big) \\
&= \frac{i}{2}\,\mathrm{Tr\,Log}\Big(\big(G + 2g^{\eta\eta}\mu\,\partial_\eta\big)\big(G - 2g^{\eta\eta}\mu\,\partial_\eta\big)\Big) \\
&= \frac{i}{2}\,V\int\frac{dk^3}{(2\pi)^3}\Big(\mathrm{Log}\big(\partial_\eta^2 + \gamma^{-1}g^{ij}\partial_i(\gamma\,\partial_j) + M^2 + 2ig^{\eta\eta}\mu\,\partial_\eta\big) \\
&\qquad + \mathrm{Log}\big(\partial_\eta^2 + \gamma^{-1}g^{ij}\partial_i(\gamma\,\partial_j) + M^2 - 2ig^{\eta\eta}\mu\,\partial_\eta\big)\Big),
\end{aligned} \tag{25}
$$

where $M^2 \equiv m^2 - g^{\eta\eta}\mu^2$ and $V \equiv \int d^3x\,\gamma$ is the volume for the $(\eta, x_\perp)$ space–time, which appears from the rewriting of a functional trace into an integration:

$$
\mathrm{Tr} \to \left(\frac{L}{2\pi}\right)^3\left(\frac{2\pi}{L}\right)^3\sum_k \equiv V\int\frac{dk^3}{(2\pi)^3} \tag{26}
$$

with $\left(\frac{2\pi}{L}\right)^3\sum_k = \int dk^3$ and $V \equiv L^3 = \int d^3x\,\gamma$, where L means just length in each space for the η, $x_\perp$-directions, and $\sqrt{-\det g_{ab}} = \sqrt{-g_{\rho\rho}\det g_{ab}} = \gamma$ ($a, b = \eta, x_\perp$ except for ρ, and $g_{\rho\rho} = 1$). $d^3k \equiv d\omega\, d^2k_\perp$ with $k_\perp \equiv (k_y, k_z)$, which are momenta corresponding to the coordinate $(\eta, x_\perp)$ as in eq. (32).

Defining $\delta_\pm \equiv g^{\eta\eta}\partial_\eta^2 + \gamma^{-1}g^{ij}\partial_i(\gamma\,\partial_j) \pm 2ig^{\eta\eta}\mu\,\partial_\eta$, we further rewrite W as

$$
\begin{aligned}
W &= \frac{i}{2}\,V\int\frac{dk^3}{(2\pi)^3}\Big(\mathrm{Log}\big(\delta_+ + M^2\big) + \mathrm{Log}\big(\delta_- + M^2\big)\Big) \\
&= \frac{i}{2}\,V\int\frac{dk^3}{(2\pi)^3}\Big(\int_0^{M^2} d\Delta^2(\delta_+ + \Delta^2)^{-1} + \mathrm{Log}\,\delta_+ \\
&\qquad + \int_0^{M^2} d\Delta^2(\delta_- + \Delta^2)^{-1} + \mathrm{Log}\,\delta_-\Big).
\end{aligned} \tag{27}
$$

Here, we can ignore $\int \frac{dk^3}{(2\pi)^3}\,\mathrm{Log}\,\delta_\pm$ for the reason in what follows: First, since we will perform the derivative with regard to the chemical potential to obtain the particle density at the end, we look at only the part concerning the chemical potential as

$$
\begin{aligned}
&\int\frac{dk^3}{(2\pi)^3}\,\mathrm{Log}\,\delta_\pm \\
&= \int\frac{dk^3}{(2\pi)^3}\,\mathrm{Log}\,\big(\partial_\eta^2 + \gamma^{-1}g^{ij}\partial_i(\gamma\,\partial_j)\big) \\
&\quad + \int dk_\eta\frac{dk_\perp^2}{(2\pi)^3}\,\mathrm{Log}\left(1 \pm \frac{2ig^{\eta\eta}\mu\,\partial_\eta}{\partial_\eta^2 + \gamma^{-1}g^{ij}\,\partial_i(\gamma\,\partial_j)}\right) \\
&\sim \int dk_\eta\frac{dk_\perp^2}{(2\pi)^3}\,\mathrm{Log}\left(1 \pm \frac{2ig^{\eta\eta}\mu\,\partial_\eta}{\partial_\eta^2 + \gamma^{-1}g^{ij}\,\partial_i(\gamma\,\partial_j)}\right).
\end{aligned} \tag{28}
$$

In the above, we have described the integral $\int dk^3$ separately as $\int dk_\eta \int dk_\perp^2$. Then, replacing ∂_i with $i\,k_i$, from the fact that $\int dk\,k\log(1 + \frac{c_1 k}{c_2 + k^2}) = 0$ ($c_{1,2}$ are some constants), we can see that the μ-dependent part vanishes in the $k_\perp$-integrals.

Finally, we can write the effective action W in eq. (27) as

$$
W = \frac{i}{2}V\,\gamma\int\frac{dk^3}{(2\pi)^3}\int_0^{M^2} d\Delta^2\,(\widetilde{D}_+ + \widetilde{D}_-), \tag{29}
$$

where $\widetilde{D}_\pm$ is

$$
\gamma^{-1}\left(\delta_\pm + \Delta^2\right)^{-1} \equiv \widetilde{D}_\pm = \widetilde{D}_\pm(k_\eta, \rho, k_\perp). \tag{30}
$$

In the above, we wrote the arguments concerning the momenta as k_η and $k_\perp$ despite that these are given by the differential operators in the actual expression, which may be allowed.

3.2. Calculation of $\widetilde{D}_\pm$

From the definition in eq. (30), we can have the following equation:

$$
\begin{aligned}
\delta^4(x - x') &= \gamma\left(\delta_\pm + \Delta^2\right)D_\pm(x - x') \\
&= \gamma\left(g^{\eta\eta}\partial_\eta^2 + \gamma^{-1}g^{ij}\partial_i(\gamma\partial_j) \pm 2i\,g^{\eta\eta}\mu\,\partial_\eta + \Delta^2\right) \\
&\quad \times D_\pm(x - x')
\end{aligned} \tag{31}
$$

with $D_\pm(x - x')$ defined as

$$
D_\pm(x - x') = \int\frac{d^3k}{(2\pi)^3}\,\widetilde{D}_\pm\,e^{i\,(\omega\,(\eta - \eta') - k_\perp(x_\perp - x'_\perp))}. \tag{32}
$$

We can rewrite eq. (31) as

$$
\begin{aligned}
&\delta(\rho - \rho')\int\frac{d^3k}{(2\pi)^3}\,e^{i\,(\omega\,(\eta - \eta') - k_\perp(x_\perp - x'_\perp))} \\
&= \int\frac{d^3k}{(2\pi)^3}\,\gamma\left(g^{\eta\eta}\partial_\eta^2 + \gamma^{-1}g^{ij}\partial_i(\gamma\partial_j) \pm 2i\,g^{\eta\eta}\mu\,\partial_\eta + \Delta^2\right) \\
&\quad \times \widetilde{D}_\pm\,e^{i\,(\omega\,(\eta - \eta') - k_\perp(x_\perp - x'_\perp))} \\
&= \int\frac{d^3k}{(2\pi)^3}\,\gamma\left(\rho^{-2}\partial_\eta^2 - \rho^{-1}\partial_\rho(\rho\partial_\rho) - (\partial_x^2 + \partial_y^2)\right. \\
&\qquad \left. \pm 2i\,g^{\eta\eta}\mu\,\partial_\eta + \Delta^2\right)\widetilde{D}_\pm\,e^{i\,(\omega\,(\eta - \eta') - k_\perp(x_\perp - x'_\perp))},
\end{aligned} \tag{33}
$$

where $d^3k \equiv d\omega\,dk_\perp^2$. As a result, we can obtain the equation which $\widetilde{D}_\pm$ should satisfy as

$$
\begin{aligned}
\gamma^{-1}\delta(\rho - \rho') = &\big(-\rho^{-2}\omega^2 - \rho^{-1}\partial_\rho - \partial_\rho^2 + k_y^2 \\
&+ k_z^2 \pm 2\rho^{-2}\mu\,\omega + \Delta^2\big)\widetilde{D}_\pm.
\end{aligned} \tag{34}
$$

Finally, we can obtain the equation that determines $\widetilde{D}_\pm$ as

$$
\big(\mathcal{F} + \Omega_\pm^2\big)\,\widetilde{D}_\pm = -\rho\,\delta(\rho - \rho'), \tag{35}
$$

where $\mathcal{F} \equiv \rho^2\partial_\rho^2 + \rho\,\partial_\rho - \rho^2\kappa^2$ with $\kappa^2 \equiv k^2 + \Delta^2$ ($k^2 \equiv k_y^2 + k_z^2$), and $\Omega_\pm^2 \equiv \omega(\omega \mp 2\mu)$.

Let us now determine $\widetilde{D}_\pm$ from eq. (35). First, we can see that $\mathcal{F}$ can satisfy the following differential equation:

$$
\mathcal{F}\,\Theta_\lambda(\rho, k) = (i\lambda)^2\,\Theta_\lambda(\rho, k), \tag{36}
$$

with

$$
\Theta_\lambda(\rho, k) = C_\lambda K_{i\lambda}(\kappa\rho), \quad C_\lambda = \frac{1}{\pi}\sqrt{2\lambda\sinh(\pi\lambda)}, \tag{37}
$$

where the values of λ are positive real numbers, $K_{i\lambda}(\kappa\rho)$ is the second kind modified Bessel function, and C_λ was obtained from the equation that the second kind modified Bessel functions satisfy,

$$
\int_0^\infty \frac{d\rho}{\rho}\,\Theta_{\lambda'}(\rho, k)\,\Theta_\lambda(\rho, k) = \delta(\lambda' - \lambda). \tag{38}
$$

We then assume that $\widetilde{D}_\pm$ can be written by taking $\Theta_\lambda(\rho, k)$ as the bases for the orthogonal directions labeled by λ as

$$
\widetilde{D}_\pm = \int_0^\infty d\lambda\,f_{\lambda,\pm}(\omega, \rho')\,\Theta_\lambda(\rho, k), \tag{39}
$$

where the above means an orthogonal base expansion of $\Theta_\lambda(\rho,k)$ labeled by λ, and $f_{\lambda,\pm}(\omega,\rho')$ are the coefficients of each direction. If we can find $f_{\lambda,\pm}(\omega,\rho')$ that satisfies eq. (35), this assumption is right. We now obtain such $f_{\lambda,\pm}(\omega,\rho')$.

We can see that now we can have the following two independent equations as

$$\int \frac{d\rho}{\rho}\,\widetilde{D}_\pm\,\mathcal{F}\,\Theta_\lambda(\rho,k) = -\lambda^2 \int \frac{d\rho}{\rho}\,\widetilde{D}_\pm\,\Theta_\lambda(\rho,k), \tag{40a}$$

$$\int \frac{d\rho}{\rho}\,\Theta_\lambda(\rho,k)\,\mathcal{F}\widetilde{D}_\pm = -\Omega_\pm^2 \int \frac{d\rho}{\rho}\,\Theta_\lambda(\rho,k)\,\widetilde{D}_\pm - \Theta_\lambda(\rho',k). \tag{40b}$$

Here, in obtaining eqs. (40a) and (40b), we have used eqs. (36) and (35), respectively.

Eq. (40a) and eq. (40b) are equivalent to each other. Actually, this equivalence can be seen easily by what eq. (40a) and eq. (40b) can be represented as $\langle \widetilde{D}_\pm | \mathcal{F} | \Theta_\lambda(\rho,k)\rangle$ and $\langle \Theta_\lambda(\rho,k) | \mathcal{F} | \widetilde{D}_\pm\rangle$, respectively. Then, by subtracting these two equations each other,

$$\begin{aligned}
0 &= \left(-\lambda^2+\Omega_\pm^2\right) \int \frac{d\rho}{\rho}\widetilde{D}_\pm\,\Theta_\lambda(\rho,k) + \Theta_\lambda(\rho',k) \\
&= \left(-\lambda^2+\Omega_\pm^2\right) \int \frac{d\rho}{\rho}\cdot \int d\lambda'\, f_{\lambda',\pm}(\omega,\rho')\Theta_{\lambda'}(\rho,k)\cdot\Theta_\lambda(\rho,k) \\
&\quad + \Theta_\lambda(\rho',k) \\
&= \left(-\lambda^2+\Omega_\pm^2\right) \int d\lambda' \cdot \int \frac{d\rho}{\rho}\,\Theta_{\lambda'}(\rho,k)\Theta_\lambda(\rho,k)\cdot f_{\lambda',\pm}(\omega,\rho') \\
&\quad + \Theta_\lambda(\rho',k) \\
&= \left(-\lambda^2+\Omega_\pm^2\right) \int d\lambda' \cdot \delta(\lambda'-\lambda)\cdot f_{\lambda',\pm}(\omega,\rho') + \Theta_\lambda(\rho',k) \\
&= \left(-\lambda^2+\Omega_\pm^2\right) f_{\lambda,\pm}(\omega,\rho') + \Theta_\lambda(\rho',k).
\end{aligned} \tag{41}$$

In the above, we have used eqs. (38) and (39). From the above, $f_{\lambda,\pm}(\omega,\rho')$ can be determined as

$$f_{\lambda,\pm}(\omega,\rho') = \frac{\Theta_\lambda(\rho',k)}{\lambda^2-\Omega_\pm^2} \equiv \frac{\Theta_\lambda(\rho',k)}{\lambda^2-\Omega^2}. \tag{42}$$

In the above, we have regarded Ω_+^2 and Ω_-^2 as the same each other, and written as $\Omega^2 \equiv \Omega_+^2 = \Omega_-^2$. Because the integrations of $f_{\lambda,+}(\omega,\rho')$ and $f_{\lambda,-}(\omega,\rho')$ with regard to ω in eq. (32) do not produce any difference under the transformation $\omega \to -\omega$. Namely,

$$\begin{aligned}
& D_+(x-x') \\
&= \int \frac{d\omega\, d^2k_\perp}{(2\pi)^3}\cdot \int d\lambda\, \frac{\Theta_\lambda(\rho',k)\,\Theta_\lambda(\rho,k)}{\lambda^2-\omega(\omega-2\mu)}\, e^{i\left(\omega(\eta-\eta')-k_\perp(x_\perp-x'_\perp)\right)} \\
&= \int \frac{d\omega\, d^2k_\perp}{(2\pi)^3}\cdot \int d\lambda\, \frac{\Theta_\lambda(\rho',k)\,\Theta_\lambda(\rho,k)}{\lambda^2-\omega(\omega+2\mu)}\, e^{i\left(\omega(\eta-\eta')-k_\perp(x_\perp-x'_\perp)\right)} \\
&= D_-(x-x').
\end{aligned} \tag{43}$$

As a result, we can put $\Omega_\pm^2 \equiv \Omega^2$. Correspondingly, there is no difference between $\widetilde{D}_+$ and $\widetilde{D}_-$. Hence, we put $\widetilde{D}_\pm \equiv \widetilde{D}$ and $D_\pm \equiv D$.

Now, substituting eq. (42) into eq. (39), we can write $\widetilde{D}$ as

$$\widetilde{D} = \int d\lambda\, \frac{\Theta_\lambda(\rho',k)\Theta_\lambda(\rho,k)}{\lambda^2-\Omega^2}. \tag{44}$$

Finally, the effective action W can be written as

$$W = i\,V\gamma \int \frac{dk^3}{(2\pi)^3} \int_0^{M^2} d\Delta^2\, \widetilde{D}. \tag{45}$$

3.3. Calculation with the Euclidianization

Now that $\widetilde{D}_\pm = \widetilde{D}$ have been obtained, backing to eq. (29), we perform the integrations. We start with performing the Wick rotation toward the η-direction. At this time, the metric given in eq. (4) changes to

$$ds_E^2 = \rho^2 d\eta^2 + d\rho^2 + dx_\perp^2. \tag{46}$$

We can see from this form that the η-direction is S^1-compactified with the period $\beta = 2\pi$.

Then, the following replacement arises:

$$V \equiv \int d^3x\,\gamma \to -i\beta \int d^2x_\perp\,\gamma_E \equiv -i\beta\, V_E \tag{47}$$

with

$$\gamma \equiv \sqrt{-g} \to \gamma_E \equiv \sqrt{g} = \rho, \tag{48}$$

$$\int d\omega \to \frac{2\pi i}{\beta}\sum_n \quad \text{and} \quad \omega \to \omega_n = n, \tag{49}$$

where V is defined first under eq. (26), and the reason for $\omega_n = n$ is that, generally, ω_n in the S^1-compactification with a period R is given as $\omega_n = 2\pi n/R$, and in our situation $R = 2\pi$. Correspondingly, the values Ω takes are discretized, and we change the notation of Ω as $\Omega \to \Omega_n$.

As a result, the effective potential Γ defined as $W_E \equiv \beta V_E \Gamma$ with $\exp(iW) \equiv \exp(-W_E)$ can be obtained as

$$\begin{aligned}
\Gamma &= \frac{\rho}{\pi^3}\sum_{n=-\infty}^{\infty}\int \frac{d^2k_\perp}{(2\pi)^2}\int_0^{M^2} d\Delta^2 \int d\lambda\,\lambda\,\sinh(\pi\lambda)\,\frac{K_{i\lambda}^2(\kappa\rho)}{\lambda^2-\Omega_n^2} \\
&= \frac{\rho}{2\pi^3}\int d\lambda\,\lambda\,\sinh(\pi\lambda)\,\Phi \int dk\,k\Psi.
\end{aligned} \tag{50}$$

In the above, $d^2k_\perp = 2\pi k dk$ $(0 \le k \le \infty)$,

$$\Psi \equiv \int_0^{M^2} d\Delta^2\, K_{i\lambda}^2(\kappa\rho), \tag{51}$$

and we have evaluated the summation of n as

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{\lambda^2-\Omega_n^2} &= \frac{\pi}{\sqrt{\lambda^2+\mu^2}}\Big(\coth\left(\pi(\mu-\sqrt{\lambda^2+\mu^2})\right) \\
&\qquad - \coth\left(\pi(\mu+\sqrt{\lambda^2+\mu^2})\right)\Big) \\
&\equiv \pi\,\Phi.
\end{aligned} \tag{52}$$

Next, since we finally calculate the critical acceleration of the Bose–Einstein condensation, we focus on the critical moment. The value of the chemical potential at the critical moment is given by m/a_c, where a_c is the critical acceleration. We explain this in what follows.

3.4. The chemical potential at the critical moment

Using the effective potential Γ given in eq. (50), the particle density $d = -\partial\Gamma/\partial\mu$ can be written as

$$d = -\frac{\rho}{2\pi^3}\int d\lambda\,\lambda\,\sinh(\pi\lambda)\left(\frac{\partial\Phi}{\partial\mu}\int dkk\,\Psi + \Phi\int dkk\,\frac{\partial\Psi}{\partial\mu}\right). \tag{53}$$

Here, to perform $\frac{\partial\Psi}{\partial\mu}$, we show the μ-dependence of Ψ by expanded it around $\mu = 0$ as

$$\Psi = \Psi\Big|_{\mu=0} + \frac{\partial \Psi}{\partial \mu}\Big|_{\mu=0} \mu + \frac{1}{2}\frac{\partial^2 \Psi}{\partial \mu^2}\Big|_{\mu=0} \mu^2 + \cdots$$
$$= \int_0^{m^2} d\Delta\, K_{i\lambda}^2(\kappa\rho) - a^2 K_{i\lambda}^2(\kappa\rho)\Big|_{\Delta^2=m^2} \mu^2 + \cdots \tag{54}$$

with

$$\frac{\partial \Psi}{\partial \mu}\Big|_{\mu=0} = \frac{\partial M^2}{\partial \mu}\frac{\partial \Psi}{\partial M^2}\Big|_{\mu=0} = 0, \tag{55a}$$
$$\frac{1}{2}\frac{\partial^2 \Psi}{\partial \mu^2}\Big|_{\mu=0} = \frac{1}{2}\frac{\partial}{\partial \mu}\left(\frac{\partial M^2}{\partial \mu}\frac{\partial \Psi}{\partial M^2}\right)\Big|_{\mu=0} = -a^2 \frac{\partial \Psi}{\partial M^2}\Big|_{\Delta^2=m^2}. \tag{55b}$$

Using the above expansion, we can write the particle density as

$$d = -\frac{\rho}{2\pi^3}\int d\lambda\, \lambda\, \sinh(\pi\lambda) \times \left(\frac{\partial \Phi}{\partial \mu}\int dk k\, \Psi - 2a^2 \mu\, \Phi K_{i\lambda}^2(\kappa\rho)\Big|_{\Delta^2=m^2}\right). \tag{56}$$

We can always confirm numerically that $\frac{\partial\Phi}{\partial\mu} < 0$ and $\Phi > 0$. Further, we can see $\Psi > 0$, since the integrand in Ψ is given by the square and the integral direction is positive, as can be seen in eq. (51).

Hence, looking at eq. (56), we can say the following two things: When the chemical potential is increased with fixing all the other parameters, the particle density always grows up. On the other hand, when the acceleration is decreased with fixing all the other parameters, the particle density always decreased.

Hence, when decreasing the acceleration from higher accelerations, where there is no condensation, say $\alpha = 0$, with keeping the particle density to constant, the chemical potential should grow up to keep the particle density to constant. However, as can be seen from eq. (22), there is the upper bound for the value the chemical potential can take, which is m/a. Hence, when the chemical potential reaches the upper bound in such a situation, α should start to grow up, where α plays the role of the condensation as written under eq. (17).

Hence, we can see that the value of the chemical potential at the critical moment is m/a_c.

3.5. Calculation in the relativistic situation

As mentioned under eq. (52), our analysis is the one at the critical moment. Hence, for the reason mentioned above, m/a_c is assigned to the value of the chemical potential with $\alpha = 0$ and $\rho = a_c^{-1}$.

We put an assumption that m/a_c is small. This means that the chemical potential in this study is also small. The situation that m/a_c is small corresponds to the relativistic situation as mentioned in the introduction.

In what follows, we use two symbols: μ, μ'. μ means the true chemical potential, where "true" means that it has been given at eq. (6) and the derivative with regard to μ can act on. On the other hand, μ' means just a value m/a_c. μ' is not the chemical potential and the derivative cannot act on.

Then, it turns out that we can expand Φ and Ψ with regard to μ and μ' as

$$\Phi = \Phi_0 + \Phi_2\, \mu^2 + \cdots, \tag{57a}$$
$$\Psi = \Psi_0\, \mu'^2 + \Psi_2\, \mu^2 + \cdots, \tag{57b}$$

where

$$\Phi_0 \equiv \frac{2}{\lambda}\coth(\pi\lambda), \tag{58a}$$
$$\Phi_2 \equiv -\frac{1}{\lambda^3}\left(\coth(\pi\lambda) + \frac{\pi\lambda}{\sinh^2(\pi\lambda)}\big(1 - 2\pi\lambda\coth(\pi\lambda)\big)\right), \tag{58b}$$
$$\Psi_0 \equiv -\Psi_2 \equiv a_c^2\, K_{i\lambda}(k/a_c). \tag{58c}$$

At this time, substituting (57a) and (57b) into eq. (50), Γ at the critical moment in the relativistic situation is given as

$$\Gamma = \frac{1}{2a_c\pi^3}\int d\lambda\, \lambda \sinh(\pi\lambda)\int dk\, k\left(\Phi_0\Psi_0\, \mu'^2 + \Phi_0\Psi_2\, \mu^2\right). \tag{59}$$

4. The particle density and critical acceleration

We now obtain the particle density $d = -\partial\Gamma/\partial\mu$ using eq. (59) as

$$d = -\frac{\mu}{a_c\pi^3}\int_0^\infty d\lambda\lambda \sinh(\pi\lambda)\int dk k\, \Phi_0\Psi_2$$
$$= \frac{2\, a_c\, \mu}{\pi^3}\int_0^\infty d\lambda\, \cosh(\pi\lambda)\int dk k\, K_{i\lambda}^2(k/a_c)$$
$$= \frac{a_c^3\, \mu}{\pi^2}\int d\lambda\, \lambda\, \coth(\pi\lambda), \tag{60}$$

where, in the above calculation, we have used $\sinh(\pi\lambda)\,\Phi_0 = \frac{2}{\lambda}\cosh(\pi\lambda)$, and performed the integration with regard to k as $\int dk k\, K_{i\lambda}^2(k/a) = \frac{\pi\lambda}{2}\, a_c^2\, \mathrm{csch}(\pi\lambda)$.

Writing $\coth(\pi\lambda) = \frac{1}{2} + \frac{1}{e^{2\pi\lambda}-1}$ in eq. (60), we can see that the contribution from the 1/2 diverges. A similar divergence appeared in a similar scalar field model with ours [17], which was canceled by a mass renormalization. There also appeared the divergence in the Nambu–Jona–Lasinio model in the accelerating frame [15,16], which was contained by a cutoff. We may be also able to show that the 1/2 can be canceled by some regularization. However there are some unclear points in its physical meaning. The source of the divergence can be thought clearly as the ultra divergence in the one-loop order. Because our analysis can be regraded as an one-loop calculation from eq. (24), and λ can be regarded as some momentum from eq. (36). However, if we perform a mass renormalization by replacing m^2 with $m_0^2 + \delta m^2$ toward the m in the replacement of $\mu = m/a_c$ to be performed right after this as the critical moment, it seems that the counter term δm^2 becomes independent of the acceleration. In this case, the counter term determined in the accelerating frame can survive even in the inertial frame. As a result, the consistency in the link between the accelerating and inertial frames in a model is unclear. On the other hand, if we perform the regularization by using a cutoff, how to determine the cutoff concretely is unclear for us at this moment. We do not examine this divergence problem in this paper, and handle it by simply ignoring the 1/2, considering that it is canceled by some regularization.

As a result, by putting $\mu = m/a_c$ as the critical moment, we can obtain the particle density as $d = a_c^2\, m/24\pi^2$. Hence we can obtain the critical acceleration as

$$a_c = 2\pi\sqrt{\frac{6d}{m}}. \tag{61}$$

For the actual observation, we show the result in the MKS units as

$$a_c = 2\pi\sqrt{c^3\hbar\,\frac{6d}{m}} \approx 32.763\sqrt{2\pi}\,\sqrt{\frac{d}{m}}\,\left[\mathrm{cm/s^2}\right], \tag{62}$$

where d and m in the last equation are the number of particle in a centimeter unit cube and the weight of particle measured by kilogram, respectively.

5. Remark

In this paper, based on the thermal excitation brought into by the Unruh effect, we have calculated the critical acceleration at which the Bose–Einstein condensation starts to occur, as in eq. (61).

The system we have considered is an ideal gas composed of a complex scalar particle in the uniformly accelerating Minkowski space–time at zero-temperature. To this purpose, we have considered a free complex scalar field at finite density given in eq. (1) in the Rindler space given in eq. (4), and adopted an expression as in eqs. (17a) and (17b) to describe the Bose–Einstein condensation.

In our analysis, we have given the space–time integration in the action as the one without the ρ-integration to fix the acceleration and the Unruh temperature. This is different point from Refs. [15–17]. If performing the analysis like them, there will be no difference in the final result. However, it is thought that we should describe explicitly that the space–time integration in the action is the one without the ρ-integration to make clear the physical meaning.

To carry out the calculation, we have picked up the leading terms in the integrals (57). As a result, we could obtain a result as in eq. (61). This manipulation means to assume that the complex scalar field considered in this paper has the critical temperature for the Bose–Einstein condensation in the temperature region $k_B T/2 \gg mc^2$, which is the relativistic situation that the particle's thermal motion energy is much higher than its static energy at the critical moment.

We can consider the analysis in another extremal region such as the non-relativistic situation in which the particle's thermal motion energy at the critical moment is much less than the particle's static energy as $k_B T/2 \ll mc^2$, where this T is around T_c. However, the middle calculations appearing in the expansion in this limit are very complicated, and the analysis in this limit is technically difficult. We take this to a future work.

Furthermore, the problem on the divergence in eq. (60) has not been examined in this paper. We also take this to a future work.

However, as mentioned in the introduction, there are papers concluding that the Unruh effect does not contribute to the restoration of the spontaneous symmetry breaking [18]. The system in these papers is not the finite density unlike us, and their conclusion would not be applied to our study readily.

However, we can see intuitively that the situation which our study and Refs. [15–17] can lead is strange. Because, for a situation, it can be observed in different ways depending on the frame in which observer stays. For example, for an observer in an inertial frame, it can be observed as a Bose–Einstein condensation state. But for an accelerating observer, it can be observed as an normal gas. Therefore they might be right after all.

If we perform this study more, before that, it is thought that we should make clear whether their conclusion can hold or not in the finite density models like ours.

In Ref. [24], a critical temperature has been obtained in the same model with ours but the background space–time is taken to the Euclid space at finite temperature. Their result is $\sqrt{3d/m}$ as written there, and we can see that there is difference from our result in the factor. We think that this discrepancy originates in the difference between the Rindler space that has the event-horizons and the Euclid space and what our calculation has been performed only in the right wedge of the Rindler space.

We can see from our result that the critical acceleration rooted in the thermal excitation brought into by the Unruh effect has different behavior from the critical temperature obtained by the true thermal excitation in the flat space. We may expect from this that the observational data's behaviors of the critical temperatures measured in the accelerating and the flat static frames are different each other. It would be interesting, if we can find this discrepancy in the experiments.

However, on the other hand, we think at last that the significance of this study would be what we could recognize a puzzle mentioned above.

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